7.1 Integration by Parts

Goal: We will reverse the product rule.

Before we start, add these to your basic list of integrals:

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{1}{\mathrm{ax+b}}dx = \frac{1}{a}\ln|ax+b| + C$$

Ex)

$$\int \sin(5x)dx = \frac{-1}{5}\cos(5x) + C$$

CHECK

$$\int e^{3x+1}dx = \frac{1}{3}e^{3x+1} + C$$

CHECK

$$\int \frac{1}{4x+3}dx = \frac{1}{4}\ln|4x+3| + C$$

CHECK

Derivation of Integration By Parts

The product rule says:

$$u(x)v'(x) + v(x)u'(x) = \frac{d}{dx}(u(x)v(x))$$

which can be written as

$$\int u(x)v'(x)dx + \int v(x)u'(x)dx = u(x)v(x)$$

Writing this in terms of the differentials:

$dv = v'(x)dx$ and $du = u'(x)dx$

we have

$$\int u\,dv + \int v\,du = uv$$

which we rearrange to get

Integration by Parts formula:

$$\int u\,dv = uv - \int v\,du$$

$\underline{Ex)}$

$$3x \cdot \cos(x) + 3\sin(x) = \frac{d}{dx}(3x\sin(x))$$

$$\int 3x\cos(x)dx + \int 3\sin(x)dx = 3x\sin(x) + c$$

$$\Rightarrow \int 3x\cos(x)dx = 3x\sin(x) - \int 3\sin(x)dx$$

Example:

$$\int x\cos(8x)dx$$

$u = x$ $dv = \cos(8x)dx$

$du = dx$ $v = \frac{1}{8}\sin(8x)$

Step 1: Choose *u* and *dv*.

Step 2: Compute *du* and *v*.

Step 3: Use formula (and hope)

$$= \frac{1}{8}x\sin(8x) - \int \frac{1}{8}\sin(8x)dx$$

$$= \frac{1}{8}x\sin(8x) - \frac{1}{64}(-\cos(8x)) + C$$

$$= \frac{1}{8}x\sin(8x) + \frac{1}{64}\cos(8x) + C$$

CHECK!!!

$$\frac{1}{8}\sin(8x) + x\cos(8x) - \frac{1}{8}\sin(8x)$$

SAME ✓✓✓

Example:

$$\int x^2 \ln(x)\, dx$$

$u = \ln(x)$ $\quad dv = x^2 dx$

$du = \frac{1}{x} dx$ $\quad v = \frac{1}{3}x^3$

$= \frac{1}{3}x^3 \ln(x) - \int \frac{1}{3} x^2 dx$

$= \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3 + C$

CHECK!

SAME ✓✓

$x^2 \ln(x) + \frac{1}{3}x^3 \cdot \frac{1}{x} - \frac{1}{3}x^2$

Example:

$$\int_1^e x^2 \ln(x)\, dx$$

$u = \ln(x)$ $dv = x^2\, dx$

$du = \frac{1}{x} dx$ $v = \frac{1}{3}x^3$

$$\frac{1}{3}x^3 \ln(x)\Big|_1^e - \int_1^e \frac{1}{3}x^2\, dx$$

$$\left(\left(\frac{1}{3}e^3\ln(e)\right) - 0\right) - \frac{1}{9}\left(x^3\Big|_1^e\right)$$

$$\frac{1}{3}e^3 - \frac{1}{9}(e^3 - 1)$$

$$\frac{1}{3}e^3 - \frac{1}{9}e^3 + \frac{1}{9}$$

$$-\frac{2}{9}e^3 + \frac{1}{9}$$

$$\frac{1}{9}(1 - 2e)$$

NOTE: FROM PREVIOUS PAGE

$$\frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3\Big|_1^e$$

$$= \left(\frac{1}{3}e^3\ln(e) - \frac{1}{9}e^3\right) - \left(0 - \frac{1}{9}\right)$$

$$= \frac{1}{3}e^3 - \frac{1}{9}e^3 + \frac{1}{9}$$

$$= -\frac{2}{9}e^3 + \frac{1}{9}$$

$$= \frac{1}{9}(1 - 2e^3)$$

Notes:

1. The symbols *u* and *v* **never** appear in the integration. They are just locations in the formula (no variables are changing, this is not substitution).

2. *u* and *dv* completely split up the integrand. **Once you <u>chose *u*</u>, then *dv* is everything else.**

3. The goal is to make $\int v\, du$ "nicer" than $\int u\, dv$
   (a) Pick u = "something that gives a derivative that is simpler than the original u"
   (b) Pick dv = "something that you can integrate"
   (c) And hope "vdu" is something in our table!

Example:

$$\int x^2 e^{x/2}dx$$

$u = x^2 \quad dv = e^{\frac{1}{2}x}dx$

$du = 2x\,dx \quad v = 2e^{\frac{1}{2}x}$

$= 2x^2e^{\frac{1}{2}x} - \int 4xe^{\frac{1}{2}x}dx$

$u = 4x \quad dv = e^{\frac{1}{2}x}dx$

$du = 4dx \quad v = 2e^{\frac{1}{2}x}$

SAME ✓✓✓

$= 2x^2e^{\frac{1}{2}x} - \left(8xe^{\frac{1}{2}x} - \int 8e^{\frac{1}{2}x}dx\right)$

$= 2x^2e^{\frac{1}{2}x} - 8xe^{\frac{1}{2}x} + 8\int e^{\frac{1}{2}x}dx$

$= 2x^2e^{\frac{1}{2}x} - 8xe^{\frac{1}{2}x} + 16e^{\frac{1}{2}x} + C$

Check!

$4xe^{\frac{1}{2}x} + x^2e^{\frac{1}{2}x} - 8e^{\frac{1}{2}x} - 4xe^{\frac{1}{2}x} + 8e^{\frac{1}{2}x}$

Example:

$$\int e^x \cos(x)\, dx$$

$u = e^x \quad dv = \cos(x)dx$

$du = e^x dx \quad v = \sin(x)$

$= e^x \sin(x) - \int e^x \sin(x) dx$

$u = e^x \quad dv = \sin(x)dx$

$du = e^x dx \quad v = -\cos(x)$

SAME ✓

$= e^x \sin(x) - \left(-e^x \cos(x) - \int -e^x \cos(x) dx\right)$

$\Rightarrow \quad \int e^x \cos(x) = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx$

$\Rightarrow$ (up to a constant) $\quad 2\int e^x \cos(x) dx = e^x \sin(x) + e^x \cos(x) + C_0$

$\int e^x \cos(x) dx = \frac{1}{2} e^x \sin(x) + \frac{1}{2} e^x \cos(x) + C \qquad C = \frac{1}{2} C_1$

$= \frac{1}{2} e^x (\sin(x) + \cos(x)) + C$

CHECK!!!

$\frac{1}{2} e^x \cancel{\sin(x)} + \underbrace{\frac{1}{2} e^x \cos(x) + \frac{1}{2} e^x \cos(x)}_{e^x \cos(x)} - \cancel{\frac{1}{2} e^x \sin(x)}$

Example:

$$\int \sin^{-1}(x)\,dx$$

$u = \sin^{-1}(x)$ $\quad dv = dx$

$du = \frac{1}{\sqrt{1-x^2}}dx$ $\quad v = x$

$$= x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}dx$$

NOW WHAT ?!?
SUBSTITUTION!

$u = 1 - x^2$

$du = -2x\,dx$

$\frac{1}{-2x}du = dx$

$$= x\sin^{-1}(x) - \int \frac{\cancel{x}}{\sqrt{u}} \frac{1}{-2\cancel{x}}du$$

$u^{-1/2}$

$$= x\sin^{-1}(x) + \frac{1}{\cancel{2}}\,\cancel{2}\,u^{1/2} + C$$

$$= \boxed{x\sin^{-1}(x) + \sqrt{1-x^2} + C}$$

$\boxed{\text{ASIDE}}$ IF YOU FORGOT $\frac{d}{dx}(\sin^{-1}(x))$

HERE IS HOW WE DERIVED IT IN MATH 124.

$\boxed{\text{From } 124}$

$y = \sin^{-1}(x)$

$\Rightarrow \sin(y) = x$

$\frac{d}{dx}[\sin(y) = x]$

$\Rightarrow \cos(y)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)}$

Now $\cos^2(y) = 1 - \sin^2(y)$

$\Rightarrow \cos(y) = \pm\sqrt{1-\sin^2(y)}$

ON DOMAIN $-\pi/2 \le x \le \pi/2$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-(\sin(y))^2}}$

x

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$